Exploring the Conformal Constraint Equations

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1 Introduction

A model for the asymptotic structure of spacetime was suggested by Roger Penrose in [22] (see also [15] for a review of the development of these ideas) using the technique of conformal rescaling. Since the reader is by now familiar with the details of the conformal rescaling construction, only enough will be said here to fix the notation to be used in the remainder of this article. The object under study will consist of a *physical spacetime* — a smooth, time- and space-orientable Lorentz manifold $(\tilde{M}, \tilde{g})$ satisfying the vacuum Einstein equation $Ric(\tilde{g}) = 0$ and that is *asymptotically simple*. In other words, it is, conformally diffeomorphic to a Lorentz manifold (M, g), called the *unphysical spacetime*, in such a way that $\tilde{g} = \Omega^{-2} g$, where the *conformal factor* is a smooth function $\Omega : M \to \mathbb{R}_+$. In addition, the boundary of M is non-empty and is associated to points at null infinity by requiring that Ω be a *defining function* for ∂M; that is, $\Omega\big|_{\partial M} = 0$ while $\mathrm{d}\Omega$ never vanishes identically along ∂M.

The purpose of the conformal boundary is to study asymptotic properties of the physical spacetime in null directions. To this end, one could use the fact that, due to the conformal equivalence with the physical spacetime, the quantities Ω and g must satisfy the conformally rescaled version of Einstein's equation, namely that $Ric(\Omega^{-2} g) = 0$. However, this equation has the drawback that it is degenerate near the boundary of M because there $\Omega \to 0$, and is thus not ideally suited for analytic investigations of the nature of the spacetime at null infinity. One possible means of avoiding this difficulty is to use a technique developed by Friedrich [16], which aims to describe the geometry of the unphysical spacetime by means of a new, yet fully equivalent system of equations derived from the equation $Ric(\Omega^{-2} g) = 0$ that is formally regular at the boundary of the unphysical spacetime. These equations involve g, Ω and several additional quantities and are known as the *conformal Einstein equations*.

As with Einstein's equations in the physical spacetime, it is possible to attempt to solve the conformal Einstein equations in the unphysical spacetime by means of an initial value formulation, where appropriate initial data are defined on a spacelike hypersurface $\mathcal{Z}$ in M and then evolved in time. Again as in the physical spacetime, the conformal equations induce certain constraint equations on the initial data; these equations are known as the *conformal constraint equations* and consist of a complicated system of coupled nonlinear differential equations for the induced metric h and second fundamental form χ of $\mathcal{Z}$, the conformal factor restricted to $\mathcal{Z}$, and several additional quantities. A particular case of interest is when $\mathcal{Z}$ is asymptotically *hyperboloidal*, i. e. $\mathcal{Z}$ intersects ∂M transversely. In this case, the evolution of the boundary of $\mathcal{Z}$ forward in time produces the conformal boundary of the unphysical spacetime, and global questions concerning the existence of classes of spacetimes satisfying the definition of asymptotic simplicity can be addressed. See [15] or [17, 18] for a review of these ideas.

The purpose of this article is twofold. First, it is to introduce the conformal constraint equations and to investigate some of their properties, which will be done in Section 2. It will be found that, in a certain sense, they describe in a coupled way two mathematical problems — namely, the elliptic boundary value problem for the conformal factor Ω and the constraint problem arising from the Gauss-Codazzi equations of $\mathcal{Z}$. Furthermore, a simple geometric assumption will be shown to lead to a special case of the equations in which the first problem does not appear and the second is in the forefront. In this special case, the full system of conformal constraint equations reduces to a much simpler and smaller system of equations that will be called the *extended constraint equations* because they will turn out to be equivalent to the usual vacuum Einstein constraint equations satisfied by the metric and second fundamental form of $\mathcal{Z}$. (Tackling the boundary value problem is at present beyond the scope of this article but will be considered in the future.)

The second goal of this article is to set up a perturbative approach for generating solutions of the extended constraint equations in the neighbourhood of a known solution, but only in the case of time-symmetric data — the more general case will be handled in another future paper [7]. This task will be accomplished in Section 3 and the main theorem proved in this section appears on page 3.1. Because the extended constraint equations are equivalent to the usual constraint equations, the Main Theorem can be interpreted as a new way of finding solutions of these equations, and furthermore, it will turn out to be a way that is completely different from the 'classical' (i. e. Lichnerowicz-York) method of handling them. This issue will be discussed further in the Section 3.

2 The conformal constraint equations

2.1 Deriving the equations

Suppose (M, g, Ω) is an unphysical spacetime satisfying the assumptions of asymptotic simplicity and thus that the metric and conformal factor satisfy the rescaled version of Einstein's equation

$$Ric(\Omega^{-2} g) = 0\,. \tag{1}$$

This section sketches briefly how equation (1) for g and Ω leads first to the conformal Einstein equations for g, Ω and additional quantities, and then to the conformal constraint equations. Begin by expanding (1) to obtain

$$R_{\mu\nu} = -\frac{\Box\Omega}{\Omega}\, g_{\mu\nu} - \frac{2}{\Omega}\, \nabla_\mu \nabla_\nu \Omega + \frac{3\, \nabla^\lambda \Omega \nabla_\lambda \Omega}{\Omega^2}\, g_{\mu\nu}\,, \tag{2}$$

where $R_{\mu\nu}$ are the components of the Ricci tensor in the unphysical spacetime, ∇_μ is the covariant derivative of the four-metric and $\Box$ is its D'Alembertian operator. Notice that, as it is written, equation (2) contains terms with negative powers of Ω which tend to infinity near the boundary of the unphysical spacetime. Alternatively, if the equation is multiplied through by Ω^2, then the principal parts of the differential operators acting on g and Ω, would tend to zero at the boundary. Either way, equation (2) degenerates near the boundary of the unphysical spacetime, and as mentioned in the Introduction, this makes it an unwieldy choice for studying the geometry of the spacetime near null infinity.

Helmut Friedrich's procedure for obtaining a system of equations equivalent to the rescaled Einstein equations (2) and formally regular at the boundary of the unphysical spacetime can be found in several papers, see for example [16]. His derivation proceeds in the following way. Let $C_{\mu\nu\lambda\rho}$ be the Weyl tensor of the metric g and define the quantities

$$\begin{aligned} L_{\mu\nu} &= \frac{1}{2} R_{\mu\nu} - \frac{1}{12} R g_{\mu\nu} \\ S_{\mu\nu\lambda\rho} &= \Omega^{-1} C_{\mu\nu\lambda\rho} \\ \psi &= \frac{1}{4}\Box\Omega + \frac{1}{24} R\Omega\,. \end{aligned} \tag{3}$$

The tensor $S_{\mu\nu\lambda\rho}$ is smooth on ∂M because under the assumptions of asymptotic simplicity, Penrose has shown that $C_{\mu\nu\lambda\rho}$ vanishes at the boundary of M [23] (a further condition on the topology of ∂M — that ∂M admits spherical sections — is also needed, and will be assumed to hold). Then, by rephrasing (2) in terms of the quantities (3), adjoining the Bianchi identity for the curvature tensors in the physical and unphsyical spacetimes, and adjoining the well-known decomposition of the curvature tensor

$$R_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} + g_{\mu\lambda}L_{\nu\rho} - g_{\mu\rho}L_{\nu\lambda} + g_{\nu\rho}L_{\mu\lambda} - g_{\nu\lambda}L_{\mu\rho} \, ,$$

one obtains the system of equations

$$\begin{aligned} \nabla_\mu \nabla_\nu \Omega &= -\Omega L_{\mu\nu} + \psi g_{\mu\nu} \\ \nabla_\mu \psi &= -L_{\mu\nu} \nabla^\nu \Omega \\ \nabla_\lambda L_{\mu\nu} - \nabla_\mu L_{\lambda\nu} &= \nabla^\rho \Omega \, S_{\mu\lambda\nu\rho} \\ \nabla^\rho S_{\mu\lambda\nu\rho} &= 0 \\ 2\Omega\psi - \nabla_\mu \Omega \nabla^\mu \Omega &= 0 \\ R_{\mu\nu\lambda\rho} &= \Omega S_{\mu\nu\lambda\rho} + g_{\mu[\lambda}L_{\nu]\rho} - L_{\mu[\lambda}g_{\nu]\rho} \, , \end{aligned} \tag{4}$$

by means of lengthy, though straightforward algebraic manipulations. The equations above are known as the *conformal Einstein equations*.

The equivalence of (4) to (2) is confirmed as follows. Suppose the quantities L, S and ψ as well as g and Ω satisfy (4). Then by algebra, it can be shown that the pair (g, Ω) satisfies (2) and that L, S and ψ relate to Ω and the curvature quantities in the manner indicated in (3). (The algebra is fairly straightforward: for instance, the last equation in (4) identifies L and S as components of the curvature tensor; then it is a matter of computation to recover equation (2) from the remaining five.)

It is immediately clear that the equations in (4) are regular when $\Omega = 0$. Furthermore, not only do the conformal Einstein equations contain the rescaled vacuum Einstein equations, but they also contain the Bianchi identity for the curvature tensor, though expressed in the new unknowns. Thus one can consider (4) to contain *integrability conditions* since the Bianchi identity is in some sense a integrability condition for the curvature tensor — meaning that the Bianchi identity is a result of requiring second covariant derivatives to commute properly (this can best be seen explicitly by rewriting the curvature tensor in terms of the vector-valued connection 1-forms as in [6], whereby the Bianchi identity becomes an incarnation of the identity $\mathrm{d}^2 = 0$ satisfied by the exterior differential operator).

Suppose now that $\mathcal{Z}$ is a spacelike hypersurface in M. The fact that the conformal Einstein equations constrain certain initial data on $\mathcal{Z}$ can be seen by performing a $3+1$ splitting of the spacetime near $\mathcal{Z}$. Choose a frame E_a, $a = 1, 2, 3$, for the tangent space of $\mathcal{Z}$ and complete this to a frame for the unphysical spacetime by adjoining the forward-pointing unit normal vector field n of $\mathcal{Z}$. Use this frame to decompose the equations (4) into components parallel and perpendicular to $\mathcal{Z}$. The constraint equations induced by the conformal Einstein equations are those equations arising in this process in which no second normal derivatives of g or Ω, and no first normal derivatives of

L, S or ψ appear. These equations are:

$$\begin{aligned}
\nabla_a \nabla_b \Omega &= \Sigma \chi_{ab} - \Omega L_{ab} + \psi g_{ab} \\
\nabla_a \Sigma &= \chi_a^c \nabla_c \Omega - \Omega L_a \\
\nabla_a \psi &= -\nabla^b \Omega L_{ba} - \Sigma L_a \\
\nabla_a L_{bc} - \nabla_b L_{ac} &= \nabla^e \Omega S_{ecab} - \Sigma S_{cab} - (\chi_{ac} L_b - \chi_{bc} L_a) \\
\nabla_a L_b - \nabla_b L_a &= \nabla^e \Omega \, S_{eab} + \chi_a^c L_{bc} - \chi_b^c L_{ac} \\
\nabla^a \bar{S}_{abc} &= \chi_b^a S_{ac} - \chi_c^a S_{ab} \\
\nabla^a S_{ab} &= -\chi^{ac} \bar{S}_{abc} \\
0 &= 2\Omega\psi + \Sigma^2 - \|\nabla\Omega\|^2 \\
\nabla_c \chi_{ba} - \nabla_b \chi_{ca} &= \Omega \bar{S}_{abc} + g_{ab} L_c - g_{ac} L_b \\
R_{ab} &= \Omega S_{ab} + L_{ab} + \frac{1}{4} L_c^c g_{ab} - \chi_c^c \chi_{ab} + \chi_{ca} \chi_b^c
\end{aligned} \tag{5}$$

where ∇ now denotes the covariant derivative operator on $\mathcal{Z}$ corresponding to its induced metric g and R_{ab} is its Ricci curvature. The unknown quantities appearing in these equations are the initial data. They are:

- the induced metric of $\mathcal{Z}$, which is still called g (no confusion will arise because the 4-dimensional setting will not be considered further in the remainder of the this article),
- the second fundamental form χ of $\mathcal{Z}$,
- the function Ω restricted to $\mathcal{Z}$,
- the normal derivative $n(\Omega)\big|_{\mathcal{Z}}$, denoted Σ,
- the tensors $L_{ab} = E_a^\mu E_b^\nu L_{\mu\nu}$ and $L_a = n^\mu E_a^\nu L_{\mu\nu}$,
- the tensors $\bar{S}_{abc} = n^\mu E_a^\nu E_b^\lambda E_c^\rho S_{\nu\mu\lambda\rho}$ and $S_{ab} = n^\mu n^\nu E_a^\lambda E_b^\rho S_{\lambda\mu\rho\nu}$,
- and the function ψ restricted to $\mathcal{Z}$.

The equations (5) are known as the *conformal constraint equations*. The derivation of these equations will not be reproduced here — the reader is asked to consult [16] for this material. However, it is fairly easy to recognize the origin of the various terms appearing there. For example, the first two equations arise as the tangential and tangential-normal components of the first equation of (4). Furthermore, and more importantly for the sequel, the last two equations arise as the Gauss and Codazzi equations applied to the decomposition of the curvature tensor given by the last equation of (4).

NOTE: The various tensor quantities that appear in (5) possess certain symmetries as a result of their origin as components of the curvature tensor: L_{ab} is symmetric; S_{ab} is symmetric and trace-free; and $\bar{S}_{abc}$ is antisymmetric on its last two indices, satisfies the Jacobi symmetry $\bar{S}_{abc} + \bar{S}_{cab} + \bar{S}_{bca} = 0$ and is trace-free on all its indices. (Tensors with these symmetries will appear often in the sequel. Tensors of rank three that are antisymmetric on their last two indices and satisfy the Jacobi symmetry will be called Jacobi tensors for short while those which are in addition trace-free will be called traceless Jacobi tensors.) Note that even though the tensor $S_{abcd} = E_a^\mu E_b^\nu E_c^\lambda E_d^\rho S_{\mu\nu\lambda\rho}$ appears in the constraint equations, it is not a truly independent initial datum because, thanks to the symmetries of $S_{\mu\nu\lambda\rho}$, it can be written as $S_{abcd} = g_{a[c} S_{d]b} - S_{a[c} g_{d]b}$.

The system (5) is clearly exceedingly complicated because it is quasi-linear and highly coupled. However, the advantage provided by (5) is once again that it is formally regular at the boundary of $\mathcal{Z}$. For the sake of comparison, recall the interior of $\mathcal{Z}$ can be viewed as a spacelike hypersurface of the physical spacetime, and as such, satisfies the usual Einstein constraint equations there. In other words, if its induced metric is denoted by $\tilde{g}$ and its second fundamental form by $\tilde{\chi}$, then

$$\begin{aligned} \tilde{\nabla}^a \tilde{\chi}_{ab} - \tilde{\nabla}_b \tilde{\chi}_a^a &= 0 \\ \tilde{R} + (\tilde{\chi}_a^a)^2 - \tilde{\chi}^{ab}\tilde{\chi}_{ab} &= 0\,, \end{aligned} \tag{6}$$

where $\tilde{\nabla}$ is the covariant derivative operator of the metric $\tilde{g}$ and $\tilde{R}$ is its scalar curvature. These equations can be rephrased in terms of g, χ and Ω in the unphysical spacetime by conformal transformation. The necessary transformation rules are that $\tilde{g} = \Omega^{-2} g$ and also that $\tilde{\chi} = \Omega^{-1}\chi + \Sigma\Omega^{-2} g$ (which can be found by conformally transforming the definition of the second fundamental form as the normal component of the covariant derivative restricted to $\tilde{\mathcal{Z}}$). The resulting equations are

$$\begin{aligned} \Omega^2\big(R + (\chi_a^a)^2 - \chi^{ab}\chi_{ab}\big) + 4\Omega\Delta_g\Omega - 6\|\nabla\Omega\|^2 + 4\Omega\Sigma\chi_a^a + 6\Sigma^2 &= 0 \\ \Omega\big(\nabla_a \chi_b^a - \nabla_b \chi_a^a\big) - 2\nabla_b\Sigma - 2\chi_b^a\,\nabla_a\Omega &= 0\,, \end{aligned} \tag{7}$$

where $\Sigma = n(\Omega)\big|_{\mathcal{Z}}$ and Δ_g is the Laplacian of the metric g. Once again, the principal parts of these equations contain factors of Ω and thus degenerate as $\Omega \to 0$ near the boundary of $\mathcal{Z}$. This behaviour does not arise in the conformal constraint equations.

The conformal constraint equations listed in (5) are equivalent to the usual constraint equations (7) because if $(g, \chi, \Omega, \Sigma)$ solves (7) and the additional quantities S, $\bar{S}$, L and ψ are defined as indicated in (5) (e. g. the last equation defines ψ; then the first equation defines the 2-tensor L_{ab}, etc.), then straightforward computation shows that the conformal constraint equations are satisfied; furthermore, if $(g, \chi, \Omega, \Sigma, S, \bar{S}, L, \psi)$ satisfies (5), then it can be shown that $(g, \chi, \Omega, \Sigma)$ satisfies (7), and consequently, $\tilde{g}$ and $\tilde{\chi}$, given by the transformation rules above, satisfy the usual constraint equations (6). These considerations thus suggest one method for constructing solutions of the conformal constraint equations: construct any solution $(\tilde{g}, \tilde{\chi})$ of the usual constraint equations using standard techniques, choose a conformal factor, perform the transformations to the unphysical spacetime and use the conformal constraint equations to define the subsidiary quantities in terms of $(\tilde{g}, \tilde{\chi})$. Then these new quantities satisfy the conformal constraint equations.

Consequently, it is possible to *assume* the existence of initial data satisfying (5) with well-defined asymptotic properties (essentially given by the transformation rules above) and study only the time evolution of the data according to the conformal Einstein equations (4). This is the idea behind the work of Friedrich in [16] (extended in [19]), where the time evolution of suitably small initial data on an asymptotically hyperboloidal hypersurface was studied and a complete future development was found. The nature of the asymptotic structure of this class of solutions near null infinity, and in particular the relationship between the asymptotic structure of the solution and the asymptotic structure of the initial data, was then analyzed extensively by Andersson, Chruściel and Friedrich in [4] (extended by Andersson and Chruściel in [2, 3]), and was based on the rescaled Einstein equations (4) and their constraints (7). However, the problem of the vanishing of the conformal factor near the boundary of the unphysical spacetime and the resultant degeneration of these equations remains a part of the ACF methods. Thus they are not ideally suited for certain applications, in particular for implementing numerical studies of asymptotically hyperboloidal data near null infinity where the presence of negative powers of Ω can cause computational codes to crash (see [15] for details). It is for this reason that new methods for solving (5) directly, rather than through the usual constraint equations, must be developed. This question will begin to be tackled in the remainder of this article.

2.2 Reduction to the extended constraint equations

The complexity of the conformal constraint equations makes it a daunting task to attempt to develop any methods for obtaining solutions of the equations in their full generality. However, a great deal of structure is contained within these equations, and the hope is that this structure can be exploited in the search for solutions. For instance, it is possible to disentangle in some sense the equations relating to the conformal factor and its associated boundary value problem from the equations related to the Gauss-Codazzi equations of $\mathcal{Z}$ by restricting to a special case of the equations.

The special case that will be considered in the rest of this article is to assume that the conformal diffeomorphism between $\tilde{M}$ and M is the identity, and consequently that the conformal factor is trivial (i. e. $\Omega = 1$) in the unphysical spacetime. This is somewhat of a strange simplification, because it requires that the spacetime M have empty boundary (since $\Omega^{-1}(0) = \partial M$)! One would thus not find oneself in this special case in practice since the whole point of the conformal constraint equations is to study hyperboloidal initial data in a conformally rescaled spacetime that has a boundary at null infinity. Nevertheless, the simplification afforded by the assumption $\Omega = 1$ is worthwhile to consider from a mathematical point of view because it accomplishes the disentanglement described above and allow the Gauss-Codazzi-type equations within the conformal constraint equations to be studied in isolation.

To see this explicitly, one must substitute $\Omega = 1$ and $\Sigma = 0$ (which is consistent with the assumption that $\Omega = 1$ in spacetime since $\Sigma = n(\Omega)\big|_{\mathcal{Z}} = 0$ where n is the forward-pointing unit normal of $\mathcal{Z}$) into the equations (5). One first sees that L_{ab}, L_a and ψ are forced to vanish under this assumption, and then that the conformal constraint equations reduce to the following system of four coupled equations:

$$\begin{aligned} R_{ab} &= S_{ab} - \chi_c^c \chi_{ab} + \chi_a^c \chi_{cb} \\ \nabla_c \chi_{ab} - \nabla_b \chi_{ac} &= \bar{S}_{abc} \\ \nabla^a \bar{S}_{abc} &= \chi_b^a S_{ac} - \chi_c^a S_{ab} \\ \nabla^a S_{ab} &= -\chi^{ac} \bar{S}_{abc} \,. \end{aligned}$$

Here, covariant derivatives are taken with respect to the induced metric g_{ab} of $\mathcal{Z}$ and χ_{ab} is the second fundamental form of $\mathcal{Z}$. As before, the tensor S_{ab} is symmetric and trace-free with respect to g_{ab} whereas the tensor $\bar{S}_{abc}$ is a traceless Jacobi tensor. These four quantities are the unknowns for which these equations must be solved. For reasons that will become apparent later on, it will be helpful work instead with the equivalent system obtained by replacing S_{ab} and S_{ac} in the third equation by R_{ab} and R_{ac} from the first equation. The system one obtains is actually just

$$\begin{aligned} R_{ab} &= S_{ab} - \chi_c^c \chi_{ab} + \chi_a^c \chi_{cb} \\ \nabla_c \chi_{ab} - \nabla_b \chi_{ac} &= \bar{S}_{abc} \\ \nabla^a \bar{S}_{abc} &= \chi_b^a R_{ac} - \chi_c^a R_{ab} \\ \nabla^a S_{ab} &= -\chi^{ac} \bar{S}_{abc} \,, \end{aligned} \tag{8}$$

because the terms cubic in χ vanish.

Notice that because of the symmetries of S and $\bar{S}$, if the traces of the first two equations of (8) are taken, then the usual constraint equations (6) result. Furthermore, if g_{ab} and χ_{ab} satisfy the usual constraint equations and one *defines* $\bar{S}_{abc}$ and S_{ab} by the first two equations of (8) respectively, then the remaining two equations follow by straightforward algebra and the Bianchi identity. Thus equations (8) are equivalent to the usual vacuum Einstein constraint equations and for this reason are called the *extended constraint equations*.

2.3 Properties of the extended constraint equations

The extended constraint equations (8) are clearly formally much simpler than the full system of conformal constraint equations. However, several essential features of the full equations remain. These features refer to the ellipticity properties of the various differential operators appearing in (8) as well as to the integrability conditions built into these equations.

Ellipticity Properties

One must consider the principal symbols of the operators that appear on the left hand sides of the extended constraint equations in order to understand their ellipticity properties. Begin with a definition of the symbol. Recall that if $P : C^\infty(\mathbb{R}^n, \mathbb{R}^N) \to C^\infty(\mathbb{R}^n, \mathbb{R}^M)$ is a linear differential operator of order m with constant coefficients, then it can be expressed as

$$P(u) = \sum_{\alpha_1+\cdots+\alpha_n=m} \left(\sum_{i=1}^{N} b_i^{\alpha_1\cdots\alpha_n} \frac{\partial^m u^i}{(\partial x^1)^{\alpha_1}\cdots(\partial x^n)^{\alpha_n}} \right) + P_0(u)\,,$$

where P_0 is a differential operator of order less than or equal to $m-1$ and the $b_i^{\alpha_1\cdots\alpha_n}$ are elements of $\mathbb{R}^M$. The *principal symbol* of P is the family of linear maps $\sigma_\xi : \mathbb{R}^N \to \mathbb{R}^M$ given by

$$\sigma_\xi(v) = \sum_{\alpha_1+\cdots+\alpha_n=m} \left(\sum_{i=1}^{N} b_i^{\alpha_1\cdots\alpha_n} \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} v^i \right)$$

for any non-zero $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $v \in \mathbb{R}^N$. The operator P is called *underdetermined elliptic* if the symbol is surjective for each non-zero ξ, *overdetermined elliptic* if the symbol is injective for each non-zero ξ and simply *elliptic* if the symbol is bijective for each non-zero ξ. An operator with non-constant coefficients has a symbol at each point of the domain, while for a nonlinear operator, it is the linearization which has a symbol at each given $u \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$. Such operators are overdetermined, underdetermined or elliptic if their symbols possess these properties uniformly.

To understand the ellipticity properties of the conformal constraint equations, begin with the equation for the metric g_{ab}. It is quasi-linear in g, with highest-order terms given by

$$g_{ab} \mapsto g^{cd} \left(\frac{\partial^2 g_{ad}}{\partial x^b \partial x^c} + \frac{\partial^2 g_{bd}}{\partial x^a \partial x^c} - \frac{1}{2}\frac{\partial^2 g_{ab}}{\partial x^c \partial x^d} - \frac{1}{2}\frac{\partial^2 g_{cd}}{\partial x^a \partial x^b} \right).$$

The linearization of this expression at a given metric is neither over- nor underdetermined elliptic, nor is it elliptic. However, it is well known that the Ricci curvature is degenerate as an operator on metrics because it is invariant under changes of coordinates of the metric, and that the Ricci curvature operator can be made formally elliptic by making an appropriate choice of coordinate gauge. The standard choice is to require that the metric be expressed in *harmonic coordinates*, which are defined by the requirement that the coordinate functions x^a are harmonic functions, i. e. that $\Delta_h x^a = 0$ for each a. (Since the metric itself depends on the coordinate functions, the requirement that the coordinates be harmonic is in fact a nonlinear condition. Nevertheless, the existence of such coordinates, defined outside sufficiently large balls in $\mathbb{R}^3$ for any asymptotically flat metric, has been guaranteed by Bartnik in [5].)

To show that the Ricci operator is elliptic in harmonic coordinates, first note that a straightforward calculation implies that the harmonic coordinate condition $\Delta_g x^a = 0$ for all a is equivalent to the condition $g^{bc}\Gamma^a_{bc} = 0$ for all a on the Christoffel symbols of g. Now set $\Gamma^a = g^{bc}\Gamma^a_{bc}$ (and also $\Gamma_a = g_{as}\Gamma^s$), and then recall that the components of the Ricci tensor satisfy

$$R_{ab} = R^H_{ab} + \frac{1}{2}(\Gamma_{a;b} + \Gamma_{b;a}) \tag{9}$$

where R^H_{ab} are the components of the *reduced* Ricci operator defined by

$$R^H_{ab} = -\frac{1}{2} g^{rs} g_{ab,rs} + q(\Gamma)\,. \tag{10}$$

In the expressions above, a comma denotes ordinary differentiation with respect to the coordinates, a semicolon denotes covariant differentiation (since Γ^a is not a tensor, this is to be taken formally; i. e. $\Gamma_{a;b} = \Gamma_{a,b} - \Gamma_s\Gamma^s_{ab}$), and $q(\Gamma)$ denotes a term that is quadratic in the components Γ^a. The reduced Ricci operator is clearly elliptic in g. Since $\Gamma^a = 0$ for all a in harmonic coordinates, $R_{ab}(g) = R^H_{ab}(g)$ in these coordinates, and thus the Ricci operator is elliptic in g when g satisfies the harmonic coordinate condition.

The second equation in the extended constraint equations is linear in χ_{ab} and its left hand side defines a differential operator $\chi_{ab} \mapsto \nabla_c \chi_{ab} - \nabla_b \chi_{ac}$ from the space of symmetric tensors to the space of Jacobi tensors. (It can be easily verified that the left hand side of the first equation in (8) satisfies the relevant symmetries. However, it can also be verified that the left hand side is not *a priori* traceless on all its indices — this is only a requirement on the eventual solution since the left hand side is equated with a traceless Jacobi tensor.) The principal symbol of this operator is

$$\sigma_\xi : \chi_{ab} \mapsto \xi_c \chi_{ab} - \xi_b \chi_{ac} \,.$$

By the following simple argument, one can show that σ_ξ has a one-dimensional kernel and is not surjective.

Suppose first that $\sigma_\xi(\chi_{ab}) = 0$ for some non-zero ξ. Since $\xi_a \xi^a \neq 0$, one can write uniquely $\chi_{ab} = \chi^0_{ab} + c\xi_a \xi_b$ for some c, where χ^0_{ab} is trace-free. Substituting this expression for χ_{ab} yields

$$\xi_b \chi^0_{ac} - \xi_c \chi^0_{ab} = 0 \,. \tag{11}$$

Taking the trace over a and b implies that $\xi^c \chi^0_{ac} = 0$. Then, contracting with ξ^c gives $\xi^c \xi_c \chi^0_{ab} = 0$, or $\chi^0_{ab} = 0$. Consequently, the kernel of the symbol σ_ξ is one-dimensional, and consists of tensors of the form $c\xi_a \xi_b$. Next, since the space of symmetric 2-tensors is six-dimensional, the image of the symbol is five-dimensional. Now, the target space of Jacobi tensors is eight-dimensional because any Jacobi tensor can be decomposed as $T_{abc} = \varepsilon^e{}_{bc} F_{ae} + A_b g_{ac} - A_c g_{ab}$ where F_{ae} is a trace-free and symmetric tensor (accounting for five dimensions), A_b is a 1-form (accounting for the remaining three), and ε_{abc} is the fully antisymmetric permutation symbol. The symbol can thus not be surjective. Note, however, that when it is restricted to trace-free tensors, the principal symbol *is* at least injective. Consequently, the first equation of (8) is overdetermined elliptic when restricted to the space of trace-free symmetric 2-tensors.

The third and fourth equations in (8) are linear in $\bar{S}_{abc}$ and S_{ab} respectively. It can be shown that the operators $\bar{S}_{abc} \mapsto \nabla^a \bar{S}_{abc}$ and $S_{ab} \mapsto \nabla^a S_{ab}$ are underdetermined elliptic by demonstrating that their principal symbols $\bar{S}_{abc} \mapsto \xi^a \bar{S}_{abc}$ and $S_{ab} \mapsto \xi^a S_{ab}$ are surjective maps from the space of symmetric, trace-free tensors onto the space of 1-forms and from the space of traceless Jacobi tensors onto the space of antisymmetric 2-tensors, respectively. These are fairly straightforward calculations and left to the reader.

Integrability Conditions

As mentioned in Section 2.1, the conformal Einstein equations (4) satisfied by the unphysical spacetime contain the Bianchi identity which was interpreted as being a integrability condition. Integrability conditions are also to be found in the conformal constraint equations — at least in the special case $\Omega \equiv 1$. (It is also true that many integrability conditions are contained within the full equations, but these will not be explicitly demonstrated here). To exhibit these integrability conditions, begin by considering the first and fourth equations in (8). The Bianchi identity for the Ricci curvature is

$$\nabla^a R_{ab} - \frac{1}{2} \nabla_b R = 0 \,,$$

whereby the first equation of (8) implies

$$\begin{aligned} 0 &= \nabla^a \big(S_{ab} - \chi^c_c \chi_{ab} + \chi^c_a \chi_{cb} \big) - \frac{1}{2} \nabla_b \big(-(\chi^c_c)^2 + \chi^{ac}\chi_{ac} \big) \\ &= \nabla^a S_{ab} - \big(\chi^c_c \delta^a_b - \chi^a_b \big) \big(\nabla^u \chi_{au} - \nabla_a \chi^u_u \big) - \chi^{ca} \big(\nabla_b \chi_{ac} - \nabla_c \chi_{ab} \big) \\ &= \nabla^a S_{ab} - \big(\chi^c_c \delta^a_b - \chi^a_b \big) h^{uv} \bar{S}_{uav} + \chi^{ac} \bar{S}_{abc} \end{aligned} \tag{12}$$

using the second equation in (8) and its trace. By the symmetries of $\bar{S}_{abc}$, the middle term in (12) vanishes, leaving

$$0 = \nabla^a S_{ab} + \chi^{ac} \bar{S}_{abc} \,,$$

which is exactly the fourth equation of (8). Thus the fourth equation of (8) encodes the Bianchi identity, the integrability condition for the curvature.

The second and third equations of (8) also consist of a constraint equation and its integrability condition as well, but in a different sense. Recall that what a integrability condition should reflect is that second second covariant derivatives commute properly. Consider, then, the result of commuting the second covariant derivatives of the second equation of (8). Begin with

$$\bar{S}_{abc} = \nabla_c \chi_{ab} - \nabla_b \chi_{ac}$$

and compute

$$\begin{aligned} \varepsilon^{ebc}\nabla_e \bar{S}_{abc} &= 2\varepsilon^{ebc}\nabla_e\nabla_c \chi_{ab} \\ &= \varepsilon^{ebc}\big(\nabla_e\nabla_c\chi_{ab} - \nabla_c\nabla_e\chi_{ab}\big) \\ &= \varepsilon^{ebc}\big(R_{eca}{}^{s}\chi_{sb} + R_{ecb}{}^{s}\chi_{as}\big) \\ &= \varepsilon^{ebc}R_{eca}{}^{s}\chi_{sb} \end{aligned} \tag{13}$$

since the symmetries of R_{abcd} imply that $\varepsilon^{abc}R_{abcd} = 0$. Now substitute in (13) the well-known decomposition of the curvature tensor in three dimensions, namely that

$$R_{eca}{}^{s} = g_{ea}R^s_c - \delta^s_e R_{ca} + \delta^s_c R_{ea} - g_{ca}R^s_e - \frac{1}{2}R\big(g_{ea}\delta^s_c - \delta^s_e g_{ca}\big)\,,$$

to obtain

$$\varepsilon^{ebc}\nabla_e \bar{S}_{abc} = 2\varepsilon^{bc}{}_{a}\chi^s_b R_{cs}\,. \tag{14}$$

Claim: equation (14) is exactly the third equation of (8). To see this, recall that a traceless Jacobi tensor can be decomposed as $\bar{S}_{abc} = \varepsilon^{e}{}_{bc}F_{ae}$ where F_{ae} is trace-free and symmetric. Consequently,

$$\begin{aligned} \varepsilon^{ebc}\nabla_e \bar{S}_{abc} &= \varepsilon^{ebc}\nabla_e \varepsilon^{u}{}_{bc}F_{au} \\ &= 2\nabla^e F_{ae} \\ &= 2\nabla^e F_{ea} && \text{(by symmetry)} \\ &= \varepsilon_a{}^{bc}\nabla^e \varepsilon^{u}{}_{bc}F_{eu} \\ &= \varepsilon_a{}^{bc}\nabla^e \bar{S}_{ebc}\,. \end{aligned} \tag{15}$$

Thus (15) together with (14) implies that

$$\varepsilon_a{}^{bc}\nabla^e \bar{S}_{ebc} = 2\varepsilon^{bc}{}_{a}\chi^s_b R_{cs}\,,$$

which is the third equation of (8) (or at least its dual, but this is equivalent).

3 Asymptotically flat solutions of the extended constraint equations in the time symmetric case

3.1 Statement of the main theorem

Because the conformal boundary of the spacetime $\tilde{M}$ is absent under the triviality assumptions that have been made on the conformal diffeomorphism, a natural setting in which to investigate the extended constraint equations (8) is the case in which $\tilde{M}$ is asymptotically Minkowski space and that $\mathcal{Z}$ is asymptotically flat. In fact, one solution of the extended constraint equations satisfying these conditions is when $\mathcal{Z} = \mathbb{R}^3$ and the initial data is the Euclidean metric $g = \delta$ with vanishing tensors χ, S and $\bar{S}$. Neighbouring asymptotically flat solutions are those whose metric g is a small perturbation of δ that decays suitably to δ near infinity, and χ, $\bar{S}$ and S are also small and decay suitably. These solutions are in addition called *time symmetric* if their second fundamental form χ actually vanishes identically.

The theorem that will be proved in the remainder of this article is a characterization of the space of asymptotically flat *and* time-symmetric solutions of the extended constraint equations in the neighbourhood of the trivial solution given above. The case of non-time-symmetric solutions is as yet beyond the scope of this article, though a future paper by the Author will clear this up [7].

Under the assumption of time-symmetry, the requirement that $\chi = 0$ implies that $\bar{S} = 0$ as well, and so the extended constraint equations further reduce to the following system of equations:

$$\begin{aligned} \nabla^a S_{ab} &= 0 \\ R_{ab}(g) &= S_{ab} \end{aligned}$$

for the unknown metric g and unknown trace-free and symmetric tensor S. Since these equations will be solved for metrics near the Euclidean metric, it will be preferable to write metrics as small perturbations of the Euclidean metric of the form $\delta + h$ where h is a symmetric tensor suitably near 0. Thus the above system should be replaced with the system

$$\begin{aligned} \nabla^a S_{ab} &= 0 \\ R_{ab}(\delta + h) &= S_{ab} \, . \end{aligned} \tag{16}$$

The covariant derivative here corresponds to the metric $\delta + h$. The theorem that will be proved is the following.

Main Theorem: *There exists a Banach space B of free data along with a neighbourhood U of zero in B, Banach spaces Y and Y' of symmetric 2-tensors, and smooth functions $\psi : U \to Y$ and $\psi' : U \to Y'$ with $\psi(0) = \psi'(0) = 0$ so that for every $b \in U$, the following hold:*

1. *$\psi(b)$ and $\psi'(b)$ tend asymptotically towards zero;*
2. *$g \equiv \delta + \psi(b)$ defines an asymptotically flat Riemannian metric on $\mathbb{R}^3$;*
3. *$S \equiv \psi'(b)$ defines a symmetric tensor that is trace-free with respect to g;*
4. *g and S satisfy the equations* (16).

The proof of this theorem will be presented in the remaining sections of this article, and consists of essentially two steps. Since (16) is not an elliptic system (as outlined in the previous section), the first step of the proof consists of exploiting the elliptic properties of these equations to define a closely related system of equations, called the *associated system*, which is elliptic. In it, the tensor S is decomposed into a sum of two components of the form $T + \mathcal{L}^g(X)$, where T is a symmetric and trace-free tensor, X is a 1-form and $\mathcal{L}^g$ is the *conformal Killing operator* which is also the adjoint of the divergence operator $S_{ab} \mapsto \nabla^a S_{ab}$. The system (16), written in terms of this decomposition, yields equations for g, X, and T whose linearization in the g and X directions will be seen to be bijective (or near enough to being bijective — the details will be seen in due course). Thus the Implicit Function Theorem can be invoked to find solutions where the quantities g and X are expressed as functions of T. The second step is then to show that all solutions of the associated system are also solutions of the original system (16). This will turn out to be true when the metric $g = \delta + h$ is sufficiently close to δ, and relies on a Poincaré inequality and the integrability condition. The Author wishes to thank H. Friedrich for suggesting this approach for solving (16).

The method outlined above for solving the extended constraints in the time symmetric case is in fact a method for solving the usual vacuum constraint equations in the time-symmetric case (namely the equation $R(g) = 0$, which follows from (16) by taking a trace) because of the equivalence of the extended constraints and the usual constraints described earlier. The differences between this method and the 'classical' Lichnerowicz-York method for solving the constraint equations are now readily apparent. In the classical method, one freely prescribes a metric g_0 on $\mathbb{R}^3$

and considers the conformally rescaled metric $g = u^4 g_0$, where $u : \mathbb{R}^3 \to \mathbb{R}$ is an unknown function. One then reads the equation $R(u^4 g_0) = 0$ as a semi-linear elliptic equation for u. In contrast, the present method treats the metric g and the one-form X as the unknowns and leads to a quasi-linear elliptic system for these quantities in terms of the freely prescribable quantity T, which is a component of the curvature of the solution.

Remark: The Main Theorem does *not* fall into the domain of *prescribed Ricci curvature* as, for example, do the results of De Turck and his collaborators [9, 12, 13, 14]. In these papers, the authors suppose a fixed symmetric tensor S is given on a set $\mathcal{O}$ and attempt to find conditions under which a metric g exists on $\mathcal{O}$ so that $Ric(g) = S$. In the Main Theorem, by contrast, the tensor S is itself an unknown quantity and only a component is prescribed ahead of time by the free data. Furthermore, De Turck's results are local in nature since $\mathcal{O}$ is usually an open set in $\mathbb{R}^n$, while the Main Theorem of this article gives a global (though perturbative) result.

3.2 Formulating an elliptic problem

The first task in the proof of the Main Theorem is to construct the associated elliptic system that is to be solved by the Implicit Function Theorem. To this end, recall that the Ricci curvature operator in (16) is not elliptic, but that imposing the coordinate gauge choice defined by the harmonic coordinate condition makes the Ricci operator elliptic. As indicated in Section 2.3, assuming *a priori* that the harmonic coordinate condition is satisfied by the metric $\delta + h$ is equivalent to replacing the Ricci operator by the reduced Ricci operator $Ric^H(\delta + h)$. (Of course, this assumption must be justified later on; i. e. it must be shown that $\delta + h$ does indeed satisfy the harmonic coordinate condition, and this is precisely what the second step of the proof of the Main Theorem will accomplish). The remaining operator in (16) is underdetermined elliptic, and an elliptic operator can be constructed from this by using a standard technique known as the *York decomposition* (see [26] but also [8, 11] for a thorough analysis of this method). Write a symmetric, trace-free tensor S in terms of a 1-form X and a freely prescribed tensor symmetric T as

$$S(h, X, T) = T^* + \mathcal{L}^{\delta+h}(X)\,. \tag{17}$$

where $T^* = T - \frac{1}{3}\mathrm{Tr}_{\delta+h}(T)(\delta+h)$ is the trace-free part of T and $\mathcal{L}^{\delta+h}(X)$ is the conformal Killing operator with respect to the metric $\delta + h$ acting on X. This is defined for a general metric g by

$$\mathcal{L}^g_{ab}(X) = \nabla_a X_b + \nabla_b X_a - \frac{2}{3}\nabla^c X_c g_{ab}\,,$$

where ∇ is the covariant derivative of the metric g. The reason for making this choice is that the composition of the divergence operator in (16) and the conformal Killing operator, that is the composite operator $\mathrm{div}_g \circ \mathcal{L}^g$ given componentwise by

$$\begin{aligned}
\left[\mathrm{div}_g \circ \mathcal{L}^g(X)\right]_b &= \nabla^a\big(\nabla_a X_b + \nabla_b X_a - \frac{2}{3}\nabla^c X_c g_{ab}\big)\\
&= \nabla^a\nabla_a X_b + \frac{1}{3}\nabla_b\nabla^a X_a + R^a_b(g) X_a\,,
\end{aligned}$$

is elliptic, as can easily be seen by computing its symbol. However, the observation that the conformal Killing operator is the formal adjoint of the divergence operator $S_{ab} \mapsto \nabla^a S_{ab}$ taking symmetric, trace-free tensors to 1-forms obviates these calculations because it is well-known that the compositon PP^* of an underdetermined elliptic operator P with its adjoint is elliptic.

These considerations lead to the following definition of the associated system, given here in index-free notation for ease of presentation:

$$\begin{aligned}
Ric^H(\delta + h) &= S(h, X, T)\\
\mathrm{div}_{\delta+h} \circ S(h, X, T) &= 0
\end{aligned} \tag{18}$$

where $S(h, X, T)$ is as in (17) and is called the *York operator*. As will be shown in due course, the map defined by

$$\Phi(h, X, T) \equiv \begin{pmatrix} Ric^H(\delta + h) - S(h, X, T) \\ \mathrm{div}_{\delta+h} \circ S(h, X, T) \end{pmatrix} \tag{19}$$

on appropriate Banach spaces has a bounded, elliptic linearization in the h and X directions and as a result, the Implicit Function Theorem yields solutions $h(T)$ and $X(T)$ as smooth functions of sufficiently small tensors T.

3.3 Choosing the Banach spaces

Before proceeding with the solution of the equations (18), it is necessary to specify in what Banach spaces of tensors the equations are to be solved. The notion of asymptotic flatness in $\mathbb{R}^3$ should be encoded rigorously into the function spaces by requiring that the relevant objects belong to a space of tensors with built-in control at infinity. Furthermore, the spaces should be chosen to exploit the Fredholm properties of the operators appearing in the map Φ. Both these ends will be served by weighted Sobolev spaces, and an appropriate choice of these spaces for use in the Main Theorem will be made below. Begin, however, with a short introduction to these spaces.

Let T be any tensor on $\mathbb{R}^3$. (This tensor may be of any order — the norm $\|\cdot\|$ appearing in the following definition is then simply the norm on such tensors that is induced from the metric of $\mathbb{R}^3$. Also, Sobolev spaces for tensors on $\mathbb{R}^n$ can be defined equally well, but since solving the conformal constraint equations is explicitly a three-dimensional problem, all definitions and theorems concerning these spaces will be stated for $\mathbb{R}^3$.) The $H^{k,\beta}$ Sobolev norm of T is the quantity

$$\|T\|_{H^{k,\beta}} = \left(\sum_{l=0}^{k} \int_{\mathbb{R}^3} \|\nabla^l T\|^2 \sigma^{-2(\beta - l) - 3} \right)^{1/2},$$

where $\sigma(x) = (1 + r^2)^{1/2}$ is the *weight function* and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ is the squared distance to the origin. Note that Bartnik's convention for describing the weighted spaces is being used (the reason for this is psychological: if $f \in H^{k,\beta}$ and f is smooth enough to invoke the Sobolev Embedding Theorem (see below), then $f(x) = o(r^\beta)$ as $r \to \infty$, which is easy to remember — see [5] for details).

The space of $H^{k,\beta}$ functions of $\mathbb{R}^3$ will be denoted by $H^{k,\beta}(\mathbb{R}^3)$ and the space of $H^{k,\beta}$ sections of a tensor bundle B over $\mathbb{R}^3$ will be denoted by $H^{k,\beta}(B)$. As an abbreviation, or where the context makes the bundle clear, such a space may be indicated simply by $H^{k,\beta}$. Note also that the following convention for integration will be used in the rest of this paper. An integral of the form $\int_{\mathbb{R}^3} f$, as in the definition above, denotes an integral of f with respect to the standard Euclidean volume form. Integrals of quantities with respect to the volume form of a different metric will be indicated explicitly, as, for example, $\int_{\mathbb{R}^3} f \, \mathrm{dVol}_g$.

The spaces of $H^{k,\beta}$ tensors satisfy several important analytic properties and the reader is asked to consult Bartnik's paper, or others on the same topic [5, 8, 10, 11], for details. The three most important properties that will be used in the sequel are the Sobolev Embedding Theorem, the Poincaré Inequality and Rellich's Lemma; these will be restated here for easy reference.

1. The Sobolev Embedding Theorem states that if $k > \frac{3}{2}$ and T is a tensor in $H^{k,\beta}$, then T is C^0. Furthermore, if the weighted C^k_β norm of a tensor T is given by

$$\|T\|_{C^k_\beta} = \sum_{l=0}^{k} \|\nabla^l f \sigma^{-\beta + l}\|_0 \,,$$

where $\|T\|_0 = \sup\{\|T(x)\| \,:\, x \in \mathbb{R}^3\}$ (using the Euclidean metric of $\mathbb{R}^3$ to define and measure the pointwise norm of T and its derivatives), then in fact, $T \in C^0_\beta$ and $\|T\|_{C^0_\beta} \le C \|T\|_{H^{k,\beta}}$,

2. The Poincare Inequality states that if $\beta < 0$, then

$$\|f\|_{H^{0,\beta}} \leq C\|\nabla f\|_{H^{0,\beta-1}} ,$$

whenever f is a function in $H^{1,\beta}(\mathbb{R}^3)$.

3. The Rellich Lemma states that the inclusion $H^{k,\beta}(B) \subseteq H^{k',\beta'}(B)$, for any tensor bundle B, is compact when $k' < k$ and $\beta' > \beta$. In other words, if T_i is a uniformly bounded sequence of tensors in $H^{k,\beta}$, then there is a subsequence $T_{i'}$ converging to a tensor T in $H^{k',\beta'}$.

Remark: The constant C appearing in the estimates above is meant as a general numerical constant, independent of the tensors or functions measured in the estimate. In the remainder of this article, any such constant will be denoted by a generic C, unless it is important to emphasize otherwise.

In addition to the three properties above, two important results that are valid in weighted Sobolev spaces will be needed in the sequel. The first concerns integration.

Duality Lemma: *If $u \in H^{l,\gamma}(\mathbb{R}^3)$ and $v \in H^{l-2,-\gamma-3}(\mathbb{R}^3)$, then the integral $\int_{\mathbb{R}^3} u \cdot v$ is well defined. Furthermore, the functional analytic dual space of $H^{0,\gamma}(\mathbb{R}^3)$ is isomorphic to $H^{0,-\gamma-3}(\mathbb{R}^3)$ under the pairing $v \mapsto \phi_v$ where $\phi_v(u) = \int_{\mathbb{R}^3} u \cdot v$.*

Proof: Choose u and v as in the statement of the lemma. Then by Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^3} |u \cdot v| &\leq \int_{\mathbb{R}^3} |u|\sigma^{-\gamma-3/2} \cdot |v|\sigma^{-(-\gamma-3)-3/2} \\ &\leq \left(\int_{\mathbb{R}^3} u^2\sigma^{-2\gamma-3}\right)^{1/2} \left(\int_{\mathbb{R}^3} v^2\sigma^{-2(-\gamma-3)-3}\right)^{1/2} \\ &< \infty .\end{aligned}$$

The product $u \cdot v$ is thus in L^1 and so its integral is well defined. The statement about duality follows from the Riesz Representation Theorem for L^2 and the inequality above. See [20, 25] for details. □

The second result concerns the Fredholm properties of certain linear, elliptic partial differential operators on weighted Sobolev spaces.

Invertibility Theorem: *Suppose B is any tensor bundle over $\mathbb{R}^3$ and let $Q : H^{k,\beta}(B) \to H^{k-2,\beta-2}(B)$ be any linear, second order, elliptic, homogeneous, partial differential operator with constant coefficients mapping between weighted Sobolev spaces of sections of B, and $k \geq 2$. Then Q is surjective if $\beta \notin \mathbb{Z}$ and $\beta > -1$ and injective if $\beta \notin \mathbb{Z}$ and $\beta < 0$. It is thus bijective when $\beta \in (-1, 0)$. The operator Q is not Fredholm if $\beta \in \mathbb{Z}$.*

Proof: The proof of this result can be found in [10], but see also [21] for an excellent discussion of the intuitive foundation underlying the theory of elliptic operators on weighted spaces. □

Choice of Banach spaces

Denote by $S^2(\mathbb{R}^3)$ the symmetric tensors over $\mathbb{R}^3$ and by $\Lambda^1(\mathbb{R}^3)$ the 1-forms of $\mathbb{R}^3$. Let β be any number in $(-1, 0)$ and let k be any integer strictly larger than $\frac{7}{2}$. Solutions of the associated system will be found in the following Banach spaces:

- metrics $\delta + h$ will be found so that $h \in H^{k,\beta}\big(S^2(\mathbb{R}^3)\big)$;
- 1-forms X will be found in $H^{k-1,\beta-1}\big(\Lambda^1(\mathbb{R}^3)\big)$;
- tensors T will be found in $H^{k-2,\beta-2}\big(S^2(\mathbb{R}^3)\big)$.

The preceding choice of Banach spaces will be justified in the next section by showing that solutions of the associated system exist in these spaces. However, an argument can be made right now that suggests that the spaces above are indeed the correct ones in which to expect to find solutions. First, in order to ensure that the metric $\delta + h$ is asymptotically flat, h must decay as $r \to \infty$, and this holds by the Sobolev Embedding Theorem when $\beta < 0$. Next, a non-trivial, asymptotically flat metric satisfying the constraint equations must satisfy the Positive Mass Theorem [24] and consequently must have non-zero ADM mass. Thus the r^{-1} term in the asymptotic expansion of h must be allowed to be non-zero, which by the Sobolev Embedding Theorem imposes the further requirement that $\beta > -1$. Furthermore, $k \geq 4$ implies that the Sobolev Embedding Theorem can be applied to the second derivatives of the metric, and thus the curvature of the metric decays pointwise as $r \to \infty$. Finally, the h, X and T quantities are chosen in different Sobolev spaces because of the differing numbers of derivatives taken on these quantities in the associated system. For instance, the reduced Ricci curvature operator is homogeneous and of degree two and thus sends a metric in $H^{k,\beta}$ to a tensor in $H^{k-2,\beta-2}$. The operator $S(h, X, T)$ is homogeneous but is only of degree one in X and of degree zero in T; it thus maps to $H^{k-2,\beta-2}$ only when the weightings on X and T match together properly and match the weighting on the metric h as in the choice above.

3.4 First attempt to solve the associated system

The Implicit Function Theorem, the tool which will be used to solve the associated system, is restated here for ease of reference.

Implicit Function Theorem: *Let $\Phi : A \times B \to C$ be a smooth map between Banach spaces and suppose that $\Phi(0,0) = 0$. If the restricted linearized operator $D\Phi(0,0)\big|_{A\times\{0\}} : A \to C$ is an isomorphism, then there exists an open set $\mathcal{U} \subset B$ which contains 0 and a smooth function $\phi : \mathcal{U} \to A$ with $\phi(0) = 0$ so that $\Phi\big(\phi(b), b\big) = 0$.*

For an excellent discussion and proof of this theorem, see [1]. In order to use this theorem, let

$$\begin{aligned}
A &= \big\{(h, X) \in H^{k,\beta}\big(S^2(\mathbb{R}^3)\big) \times H^{k-1,\beta-1}(\Lambda^1\big(\mathbb{R}^3)\big)\big\} \\
B &= \big\{T \in H^{k-2,\beta-2}\big(S^2(\mathbb{R}^3)\big)\big\} \\
C &= H^{k-2,\beta-2}\big(S^2(\mathbb{R}^3)\big) \times H^{k-3,\beta-3}(\Lambda^1\big(\mathbb{R}^3)\big)\,;
\end{aligned}$$

then the linearization of the operator Φ in the A direction at the origin must be calculated and its mapping properties understood.

The linearization of Φ is actually quite simple when evaluated at the origin because the only nonlinearities in Φ occur in the second order terms of the reduced Ricci operator and in terms that are quadratic in the derivatives of the metric (such as in products of Christoffel symbols or in the connection terms). Since the covariant derivative of the Euclidean metric is trivial, it is thus easy to see that the linearization of a covariant derivative operator at the Euclidean metric is just the Euclidean derivative operator, and it is now a straightforward matter to deduce from the definition of the associated system in (18) that the linearization of Φ in the $A \times \{0\}$ direction is

$$\mathrm{D}\Phi(0,0,0)\,(h, X, 0) = \begin{pmatrix} -\frac{1}{2}\Delta h - \mathcal{L}(X) \\ \mathrm{div} \circ \mathcal{L}(X) \end{pmatrix}, \tag{20}$$

where Δ is the Euclidean Laplacian and $\mathcal{L}$ is the Euclidean conformal Killing operator.

Denote by P_δ the operator $\mathrm{D}\Phi(0,0,0)(\cdot,\cdot,0)$. It is a bounded linear operator between the appropriate weighted Sobolev spaces because of the way in which the weights were chosen in Section 3.3. To determine whether P_δ is an isomorphism, one appeals to the Invertibility Theorem. Recall that the weight β in the domain spaces of P_δ has been chosen between -1 and 0.

Injectivity of P_δ

Suppose (h, X) belong to the kernel of $P_\delta(h, X)$. In other words, (h, X) solves the equation $P_\delta(h, X) = (0, 0)$, or

$$-\frac{1}{2}\Delta h - \mathcal{L}(X) = 0$$
$$\operatorname{div} \circ \mathcal{L}(X) = 0\,.$$

Since the operator $\operatorname{div} \circ \mathcal{L} : H^{k-1,\beta-1}\big(\Lambda^1(\mathbb{R}^3)\big) \to H^{k-3,\beta-3}\big(\Lambda^1(\mathbb{R}^3)\big)$ is a linear, elliptic, homogeneous, constant coefficient operator of second order, the Invertibility Theorem applies, and since $\beta - 1 \in (-2, -1)$ when $\beta \in (-1, 0)$, it is thus injective. Hence $X = 0$. The remaining equation now reads $\Delta h = 0$ and again, since $\Delta : H^{k,\beta}\big(S^2(\mathbb{R}^3)\big) \to H^{k-2,\beta-2}\big(S^2(\mathbb{R}^3)\big)$ and $\beta \in (-1, 0)$, Δ is an isomorphism and thus $h = 0$. Hence P_δ is injective.

Surjectivity of P_δ

Although the operator P_δ is injective, it is *not* surjective. First note that the Invertibility Theorem does not *guarantee* surjectivity in the same way that it guaranteed injectivity. To see this, attempt to solve the equations $P_\delta(h, X) = (f, g)$ for any $f \in H^{k-2,\beta-2}\big(S^2(\mathbb{R}^3)\big)$ and $g \in H^{k-3,\beta-3}\big(\Lambda^1(\mathbb{R}^3)\big)$. In other words, consider the system of equations

$$-\frac{1}{2}\Delta h - \mathcal{L}(X) = f$$
$$\operatorname{div} \circ \mathcal{L}(X) = g\,.$$

Because $\beta - 1 \in (-2, -1)$, the operator $\operatorname{div} \circ \mathcal{L}$ is not necessarily surjective according to the Invertibility Theorem. The full equations $P_\delta(h, X) = (f, g)$ can thus not necessarily be solved.

To show that P_δ actually does fail to be surjective, it is necessary to show that the dimension of its cokernel in $H^{k,\beta}\big(S^2(\mathbb{R}^3)\big) \times H^{k-1,\beta-3}\big(\Lambda^1(\mathbb{R}^3)\big)$ is strictly greater that zero. First, note that if X_g satisfies $\operatorname{div} \circ \mathcal{L}(X_g) = g$, then the remaining equation $-\frac{1}{2}\Delta h = \mathcal{L}(X_g) + f$ *can* be solved by the Invertibility Theorem since the weight β is chosen such that Δ is an isomorphism. Thus the dimension of the cokernel of P_δ is equal to the dimension of the cokernel of $\operatorname{div} \circ \mathcal{L}$ as an operator between $H^{k-1,\beta-1}\big(\Lambda^1(\mathbb{R}^3)\big)$ and $H^{k-3,\beta-3}\big(\Lambda^1(\mathbb{R}^3)\big)$.

To characterize the cokernel of $\operatorname{div} \circ \mathcal{L}$, one appeals to general, function-theoretic properties of linear, second order, homogeneous, elliptic operators on weighted Sobolev spaces. The following lemma and its proof show how this is done.

Cokernel Lemma: *Suppose B is any tensor bundle over $\mathbb{R}^3$ and let*

$$Q : H^{k,\gamma}(B) \to H^{k-2,\gamma-2}(B)$$

be a linear, second order, homogeneous, elliptic operator on weighted Sobolev spaces of sections of B where $k \geq 2$ and $\gamma \notin \mathbb{Z}$, $\gamma < -1$. The image of the operator Q is the space:

$$\operatorname{Im}(Q) = \left\{ w \in H^{k-2,\gamma-2}(B) \;:\; \int_{\mathbb{R}^3} \langle w, z\rangle = 0 \quad \forall \quad z \in \operatorname{Ker}\left(Q^*; -1-\gamma\right) \right\}, \tag{21}$$

where the inner product $\langle \cdot, \cdot \rangle$ is induced on B from the Euclidean metric of $\mathbb{R}^3$, the operator Q^* is the formal adjoint of Q, and $\operatorname{Ker}(Q^*; -1-\gamma)$ is its kernel as an operator from $H^{k,-1-\gamma}(B)$ to $H^{k-2,-3-\gamma}(B)$.

Proof: Denote the space on the right hand side of equation (21) by W. Suppose that $k = 2$ and consider first the containment $\operatorname{Im}(Q) \subseteq W$. Choose $Q(y) \in \operatorname{Im}(Q)$ and $z \in Ker(Q^*; -1-\gamma)$. Since $Q(y) \in H^{2,\gamma-2}(B)$, the integral $\int_{\mathbb{R}^3} \langle Q(y), z\rangle$ is well defined by the Duality Lemma.
Claim: This integral equals $\int_{\mathbb{R}^3} \langle y, Q^*(z)\rangle$.

The equality of the integrals on smooth, compactly supported sections of B is true by definition of the adjoint. The equality of the integrals for $H^{k,\gamma}$ sections follows because C_c^∞ sections of B are dense in $H^{k,\gamma}$ sections of B [5].

The integral $\int_{\mathbb{R}^3} \langle Q(y), z \rangle$ is thus zero and so $Q(y) \in W$.

The reverse containment $W \subseteq \mathrm{Im}(Q)$ is proved as follows. Suppose w_0 belongs to W; thus, $w_0 \in H^{0,\gamma-2}(B)$ and satisfies $\int_{R^3} \langle w_0, z \rangle = 0$ for all $z \in Ker(Q^*; -1-\gamma)$. Suppose also that $w_0 \notin \mathrm{Im}(Q)$. Since Q is elliptic, $\mathrm{Im}(Q)$ is closed; thus by the Hahn-Banach theorem, there exists a linear functional ϕ on $H^{0,\gamma-2}(B)$ so that $\phi(w_0) \neq 0$ but $\phi\big|_{\mathrm{Im}(Q)} = 0$. Again by the Duality Lemma, there is a unique $z_0 \in H^{0,-1-\gamma}(B)$ so that $\phi(w) = \int_{\mathbb{R}^3} \langle w, z_0 \rangle$ for all $w \in H^{0,\gamma-2}(B)$. Therefore, $\phi\big|_{\mathrm{Im}(Q)} = 0$ implies that

$$
\begin{aligned}
0 &= \phi(Q(y)) \\
&= \int_{\mathbb{R}^3} \langle z_0, Q(y) \rangle \\
&= \int_{\mathbb{R}^3} \langle Q^*(z_0), y \rangle
\end{aligned}
$$

for all $y \in H^{2,\gamma}(B)$. Thus $Q^*(z_0) = 0$ or $z_0 \in \mathrm{Ker}(Q^*; -1-\gamma)$. But now, the assumptions $\phi(w_0) \neq 0$ and $\int_{\mathbb{R}^3} \langle w_0, z \rangle = 0$ for all $z \in \mathrm{Ker}(Q^*; -1-\gamma)$ are mutually contradictory. Thus it must be that $w_0 \in \mathrm{Im}(Q)$. Finally, the extension to $k > 2$ follows in a similar manner by standard functional analysis. □

Apply this theorem to the operator $Q = \mathrm{div} \circ \mathcal{L}$ with $\gamma = \beta - 1$. Now, $Q^* = Q$, so in order to solve the equation $\mathrm{div} \circ \mathcal{L}(X) = g$, the tensors g must satisfy the constraints

$$
\int_{\mathbb{R}^3} g_a Y^a = 0 \, ,
$$

where Y is any tensor in the kernel of the operator $\mathrm{div} \circ \mathcal{L}$ in the space $H^{k-1,-1-\gamma}\big(\Lambda^1(\mathbb{R}^3)\big)$.

The kernel of $\mathrm{div} \circ \mathcal{L}$ is well known and consists of 1-forms dual to the the conformal Killing fields of $\mathbb{R}^3$. There are precisely ten linearly independent families of such vector fields: the translation vector fields, the rotation vector fields, the dilation field and three so-called *special* conformal Killing fields (these correspond to transformations of the form $i \circ T \circ i$, where i is the inversion with respect to the unit circle and T is a translation). The asymptotic behaviour of these vector fields can thus be computed exactly: the translations have constant norm, the rotations and dilations have norm growing linearly in the distance from the origin, and the special vector fields have quadratic growth in the distance from the origin. Since $-1-\gamma \in (0,1)$ when $\beta \in (-1,0)$, the only 1-forms dual to the conformal Killing fields in $H^{k-1,-1-\gamma}\big(\Lambda^1(\mathbb{R}^3)\big)$ are thus those spanned by the translation 1-forms $\mathrm{d}x^1$, $\mathrm{d}x^2$ and $\mathrm{d}x^3$. Consequently, the image of $Q = \mathrm{div} \circ \mathcal{L}$ in the space $H^{k-3,\gamma-2}\big(\Lambda^1(\mathbb{R}^3)\big)$ can be characterized as follows:

$$
\mathrm{Im}(\mathrm{div} \circ \mathcal{L}) = \left\{ g \in H^{k-3,\beta-3}\big(\Lambda^1(\mathbb{R}^3)\big) \; : \; \int_{\mathbb{R}^3} g_a = 0 \, , \; a = 1,2,3 \right\} ,
$$

where g_a are the components of g in the standard coordinates of $\mathbb{R}^3$.

The conclusion that can be drawn from the analysis of this section is that the equation $\Phi(h, X, T) = (0,0)$ is *not* solvable near $(0,0,0)$ using the Implicit Function Theorem. The non-surjectivity of the linearized operator at $(0,0,0)$ is the essential obstruction. The best that can be achieved using the Implicit Function Theorem is thus that the equation $\Phi(h, X, T) = (0,0)$ can be solved *up to* a term that is transverse to the space $\mathrm{Im}(\mathrm{div} \circ \mathcal{L})$. It will turn out that this is nevertheless sufficient for solving the full equations as a result of the integrability conditions built into the equations. But in order to show this, the associated system defined in the previous section must be modified somewhat.

3.5 Reestablishing surjectivity and solving the associated system

In order to modify the associated system appropriately, first note that the space $H^{k-3,\beta-3}\big(\Lambda^1(\mathbb{R}^3)\big)$ can be written as $\mathrm{Im}(\mathrm{div}\circ\mathcal{L})\oplus W$ in many different ways; but in each case, W is a three dimensional subspace of $H^{k-3,\beta-3}\big(\Lambda^1(\mathbb{R}^3)\big)$ whose members do not integrate to zero upon taking the Euclidean inner product with the translation 1-forms. One such choice is

$$W = \mathrm{span}\,\{\phi\,\mathrm{d}x^a\}_{a=1,2,3}\ ,$$

where ϕ is any smooth, positive function of compact support whose integral over $\mathbb{R}^3$ is equal to 1.

Again, denote the domain space of the operator Φ by A. The previous paragraph suggests that one should attempt to construct a new associated operator Φ' that extends Φ in such a way that $\Phi' : A\times\mathbb{R}^3 \to \mathrm{Im}(P_\delta)\oplus W$, where the additional $\mathbb{R}^3$ factor in the domain should map under the linearization $\mathrm{D}\Phi'$ at the solution $(0,0,0;0)\in A\times\mathbb{R}^3$ onto the W factor in the image. If such a construction is possible, then the equation $\Phi'(h,X,T;\lambda)=(0,0)$ can be solved using the Implicit Function Theorem.

Construct the operator $\Phi' : A\times\mathbb{R}^3 \to H^{k-3,\beta-3}\big(\Lambda^1(\mathbb{R}^3)\big)$ according to the prescription

$$\Phi'(h,X,T;\lambda) = \begin{pmatrix} Ric^H(\delta+h) - S(h,X,T) \\ \mathrm{div}_{\delta+h}\circ S(h,X,T) - \sum_{a=1}^3 \lambda_a\phi\,\mathrm{d}x^a \end{pmatrix}, \tag{22}$$

where, as before, Ric^H is the reduced Ricci operator and $S(\cdot,\cdot,\cdot)$ is the York operator. The linearization of Φ' at $(0,0,0;0)$ in the directions transverse to the T direction is easily seen to be

$$\mathrm{D}\Phi'(\delta,0,0;0)(h,X,0;\lambda) = \begin{pmatrix} -\frac{1}{2}\Delta h - \mathcal{L}(X) \\ \mathrm{div}\circ\mathcal{L}(X) - \sum_{a=1}^3 \lambda_a\phi\,\mathrm{d}x^a \end{pmatrix}. \tag{23}$$

Denote this new operator by P'_δ. It is still bounded because ϕ has compact support, and it is now also bijective by the following arguments.

Injectivity of P'_δ

Suppose $P'_\delta(h,X;\lambda)=(0,0)$. Integrate the components of the second equation; by the divergence theorem for the Euclidean metric (valid because constant functions can be integrated against $H^{k-3,\beta-3}$ functions when $\beta\in(-1,0)$ according to the Duality Lemma), the divergence terms integrate to zero, yielding $\lambda_a=0$ for all a. The argument that both X and h are then equal to zero follows as in Section 3.4.

Surjectivity of P'_δ

Suppose that $P'_\delta(h,X;\lambda)=(f,g)$. First choose the components λ_a so that

$$\int_{\mathbb{R}^3}\big(g_a+\lambda_a\phi\big)=0$$

for each a. The equation $\mathrm{div}\circ\mathcal{L}(X) = g-\sum_{a=1}^3\lambda_a\phi\,\mathrm{d}x^a$ can then be solved for X_g according to the characterization of the image of the operator $\mathrm{div}\circ\mathcal{L}$ from the previous section. The remaining equation $-\frac{1}{2}\Delta h = \mathcal{L}(X_g)+f$ can then be solved because $\beta\in(-1,0)$ makes Δ an isomorphism.

The Implicit Function Theorem can now be invoked to solve the equation $\Phi'(h,X,T;\lambda)=(0,0)$ near $(0,0,0;0)$. To be precise, there is a neighbourhood $\mathcal{U}\subset H^{k-2,\beta-2}\big(S^2(\mathbb{R}^3)\big)$ with the following property. If $T\in\mathcal{U}$, then there is a metric $\delta+h(T)$ with $h(T)\in H^{k,\beta}\big(S^2(\mathbb{R}^3)\big)$, a covector field $X(T)\in H^{k-1,\beta-1}\big(\Lambda^1(\mathbb{R}^3)\big)$, and three real numbers $\lambda_a(T)$ so that

$$\Phi'\big(h(T),X(T),T;\lambda(T)\big)=(0,0).$$

Furthermore, the various functions $T \mapsto h(T)$, etc. are smooth in the appropriate Banach space norms. In particular, there exists a constant C so that

$$\begin{aligned} \|h\|_{H^{k,\beta}} &\leq C\|T\|_{H^{k-2,\beta-2}} \\ \|X\|_{H^{k-1,\beta-1}} &\leq C\|T\|_{H^{k-2,\beta-2}} \\ \|\lambda\|_{\mathbb{R}^3} &\leq C\|T\|_{H^{k-2,\beta-2}}\,, \end{aligned} \tag{24}$$

where $\|\cdot\|_{\mathbb{R}^3}$ denotes the standard Euclidean norm of $\mathbb{R}^3$, as long as $T \in \mathcal{U}$.

3.6 Satisfying the harmonic coordinate condition

Section 3.5 shows how the associated system (18) can be modified in such a way that it can be solved using the Implicit Function Theorem. This procedure results in a family of solutions of the equations

$$\begin{aligned} Ric^H(\delta+h) &= S(h,X,T) \\ \mathrm{div}_{\delta+h} \circ S(h,X,T) &= \lambda\phi\,, \end{aligned} \tag{25}$$

where $\lambda = \sum_{a=1}^{3} \lambda_a \, \mathrm{d}x^a$. It remains to show whether the original equations (16) are satisfied by the solution $\delta + h$ and $S(h,X,T)$. This will be done by showing that the integrability conditions built into the extended constraint equations (i. e. the Bianchi identity only, since the time-symmetric assumption has eliminated the other integrability condition) actually ensure that if $(h,X,T;\lambda)$ solves (25), then $\lambda = 0$ and $h+\delta$ satisfies the harmonic coordinate condition. Therefore $Ric^h(\delta + h) = Ric(\delta+h)$ and solutions of (25) are indeed solutions of the full equations.

To prove this claim, assume instead that λ and the quantities Γ^a are nonzero. Argue towards a contradiction as follows. First, write $g = \delta + h$ for short. The Bianchi identity $\mathrm{div}_g\big(Ric(g) - \frac{1}{2}R(g)g\big) = 0$, applied to equation (9) defining the reduced Ricci operator yields the identity

$$0 = \big(R^H_{ab} - \frac{1}{2}R^H g_{ab}\big)_{;}{}^{a} = \big(\Gamma_{a;b} + \Gamma_{b;a} - \Gamma^c{}_{;c} h_{ab}\big)_{;}{}^{a}$$

which is equivalent to

$$\Gamma_{b;a}{}^{a} + R^a_b \Gamma_a = 2\phi\lambda_a\,, \tag{26}$$

after using the modified associated system and commuting covariant derivatives appropriately. If Q_h denotes the operator $u_a \mapsto \Delta_{\delta+h} u_a + [Ric(\delta+h)]^b_a u_b$, then (26) asserts that $2\phi\lambda_a$ is in the image of $H^{k-1,\beta-1}(\Lambda^1(\mathbb{R}^3))$ under Q_h, because $h \in H^{k,\beta}\big(S^2(\mathbb{R}^3)\big)$ and the Γ^a are obtained from $\delta + h$ by differentiation. This, however, can be shown to violate the following basic result about elliptic operators.

Stability Lemma: *Let B be a tensor bundle over $\mathbb{R}^3$ and let $Q_\varepsilon : H^{l,\gamma}(B) \to H^{l-2,\gamma-2}(B)$, $\varepsilon \in [0,1]$, be a continuous family of linear, homogeneous, second order, elliptic operators, for all $\gamma \notin \mathbb{Z}$, $\gamma < -1$. Furthermore, suppose Q_ε is uniformly injective for any ε whenever $\gamma < -1$; i. e. for each $\gamma \notin \mathbb{Z}$, $\gamma < -1$, there is a constant C independent of ε so that $\|Q_\varepsilon(y)\|_{H^{l-2,\gamma-2}} \geq C\|y\|_{H^{k,\gamma}}$. If $z \notin \mathrm{Im}(Q_0)$, then there exists $\varepsilon_0 > 0$ so that $z \notin \mathrm{Im}(Q_\varepsilon)$ for all $\varepsilon < \varepsilon_0$.*

Proof: Suppose the contrary; then for some $\gamma < -1$, there exists a sequence $\varepsilon_i \to 0$ and a sequence $y_i \in H^{l,\gamma}(B)$ so that $z = Q_{\varepsilon_i}(y_i)$. By the uniform injectivity of Q_ε, $\|y_i\|_{H^{l,\gamma}} \leq C\|z\|_{H^{l-2,\gamma-2}}$ and is thus uniformly bounded. By Rellich's Lemma, there exists a subsequence $y_{i'}$ which converges to an element y in $H^{l-1,\gamma+\rho}$, where ρ is small enough so that $\gamma + \rho < -1$. Again, by uniform injectivity,

$$\begin{aligned} \|y_{i'} - y_{j'}\|_{H^{l,\gamma+\rho}} &\leq C\|Q_{\varepsilon_{i'}}(y_{i'} - y_{j'})\|_{H^{l-2,\gamma+\rho-2}} \\ &\leq C\|(Q_{i'} - Q_{j'})(y_{j'})\|_{H^{l-2,\gamma+\rho-2}} \\ &\leq C\|Q_{i'} - Q_{j'}\|_{op} \cdot \|y_{j'}\|_{H^{l,\gamma+\rho}} \end{aligned}$$

$$
\begin{aligned}
&\leq C\|Q_{i'} - Q_{j'}\|_{op} \cdot \|y_{j'}\|_{H^{l,\gamma}} \\
&\longrightarrow 0\,,
\end{aligned}
$$

by the continuity of Q_ε and the uniform boundedness of y_i. Here, $\|\cdot\|_{op}$ denotes the relevant operator norm. The subsequence $y_{i'}$ is thus Cauchy in the $H^{l,\gamma+\rho}$ norm and so $y_{i'} \to y$ in this norm. But now,

$$
z = \lim_{i' \to \infty} Q_{\varepsilon_{i'}}(y_{i'}) = Q_0(y)\,,
$$

contradicting the fact that $z \notin \mathrm{Im}(Q_0)$. □

In order to derive a contradiction from (26) using this lemma, the uniform injectivity of Q_h must be established and it must be shown that $\phi\lambda_a$ does not belong to the image of Q_0.

Uniform Injectivity of Q_h

Suppose that $Q_h(u) = 0$ for $u \in H^{k-1,\gamma}\big(\Lambda^1(\mathbb{R}^3)\big)$ where $\gamma < -1$. In other words, $\Gamma_{b;a}{}^{a} + R^a_b \Gamma_a = 0$. From this, one easily deduces

$$
-\Delta_g \|u\|^2 = 2\big(R_{ab}u^a u^b - \|\nabla u\|^2\big)\,. \tag{27}
$$

Before continuing, recall the following facts about Green's identity in weighted Sobolev spaces. If functions u and v are chosen such that $v \in H^{k,\gamma}(\mathbb{R}^3)$ and $u \in H^{k,-1-\gamma}(\mathbb{R}^3)$ for some γ, then the integrals appearing in Green's identity for a general metric g on a large ball B_r, that is

$$
\int_{B_r} u \Delta_g v \,\mathrm{dVol}_g + \int_{B_r} \nabla u \cdot \nabla v \,\mathrm{dVol}_g = \int_{\partial B_r} u \frac{\partial v}{\partial n}\,\mathrm{dA}_g\,, \tag{28}
$$

where dA_g is the area form of the metric g, are all well defined as $r \to \infty$. Thus by applying a density argument as in the proof of the Cokernel Lemma, one can conclude that

$$
\int_{\mathbb{R}^3} u \Delta_g v \,\mathrm{dVol}_g + \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \,\mathrm{dVol}_g = 0\,,
$$

in the limit of (28) as $r \to \infty$.

With this in mind, integrate both sides of equation (27) against the volume form of the metric $g = \delta + h$ to obtain

$$
-\frac{1}{2}\int_{\mathbb{R}^3} \Delta_g \|u\|^2 \,\mathrm{dVol}_g = \int_{\mathbb{R}^3} R_{ab}u^a u^b \,\mathrm{dVol}_g - \int_{\mathbb{R}^3} \|\nabla u\|^2 \,\mathrm{dVol}_g\,. \tag{29}
$$

Since $u \in H^{k-1,\gamma}$ and $1 \in H^{k-1,-\gamma-1}$ (true since $\gamma < -1$), Green's Identity applied to the left hand side of (29) gives $\int_{\mathbb{R}^3} \Delta_g \|u\|^2 = 0$. Consequently,

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^3} \|Ric(g)\|\,\|u\|^2 \,\mathrm{dVol}_g - \int_{\mathbb{R}^3} \|\nabla u\|^2 \,\mathrm{dVol}_g \\
&\leq \int_{\mathbb{R}^3} \|Ric(g)\|\,\|u\|^2 \,\mathrm{dVol}_g - C\int_{\mathbb{R}^3} \|\,\nabla \|u\|\,\|^2 \,\mathrm{dVol}_g
\end{aligned} \tag{30}
$$

for some constant C, by the Cauchy-Schwarz inequality and straightforward algebra. Next, assume that h is small in a pointwise sense (this assumption follows from the Sobolev Embedding Theorem if h is sufficiently small in the $H^{k,\beta}$ norm and $k > \frac{3}{2}$). In fact, assume that h is sufficiently close to 0 so that all norms, derivatives and volume forms of the metric g can be replaced by their Euclidean counterparts (at the expense of changing C of course). Finally, since $\|u\|$ is a scalar function, the derivative operator in (30) can be replaced by the Euclidean derivative operator without introducing lower order terms. Thus, there exists a new constant C so that the estimate

$$
0 \leq \int_{\mathbb{R}^3} \|Ric(g)\|\,\|u\|^2 - C\int_{\mathbb{R}^3} \|\,\nabla \|u\|\,\|^2 \tag{31}
$$

holds, where the norms and derivatives appearing here are those of the Euclidean metric. Next, $Ric(g) \in H^{k-2,\beta-2}$ because $g - \delta \in H^{k,\beta}$. But since $k > \frac{7}{2}$, the Sobolev Embedding Theorem gives $Ric(g) \in C^0_{-\beta+2}$. That is,

$$\sup_{\mathbb{R}^3} \left\| Ric(g) \cdot \sigma^{-\beta+2} \right\| \leq C < \infty \,,$$

which implies that

$$\sup_{\mathbb{R}^3} \left\| Ric(g) \cdot \sigma^{2} \right\| \leq C < \infty \,,$$

since $\beta < 0$. Finally, apply the Poincaré inequality for weighted Sobolev norms to the function $\|u\|$ to deduce

$$\begin{aligned} \int_{\mathbb{R}^3} \|Ric(g)\| \, \|u\|^2 &\leq \|Ric(g) \cdot \sigma^2\|_0 \int_{\mathbb{R}^3} \|u\|^2 \sigma^{-2} \\ &\leq C \|Ric(g)\|_{C^0_{-2}} \int_{\mathbb{R}^3} \| \nabla \|u\| \, \|^2 \\ &\leq C \|g - \delta\|_{C^2_0} \int_{\mathbb{R}^3} \| \nabla \|u\| \, \|^2 \\ &\leq C \|h\|_{H^{k,\beta}} \int_{\mathbb{R}^3} \| \nabla \|u\| \, \|^2 \end{aligned} \tag{32}$$

again by the Sobolev Embedding Theorem and the fact that $\beta < 0$. Using (32) in inequality (31) leads to the contradiction because the preceding estimates imply

$$0 \leq (C\|h\|_{H^{k,\beta}} - 1) \int_{\mathbb{R}^3} \| \nabla \|u\| \, \|^2 \,,$$

while if $\|h\|_{H^{k,\beta}}$ is sufficiently small, the right hand side above is clearly negative. Avoiding this contradiction requires $\nabla \|u\| = 0$. But since the Sobolev Embedding Theorem applied to $u \in H^{k-1,\gamma}$ shows that $\|u\|$ decays at infinity when $\gamma < -1$, it must be true that $u = 0$.

The operator Q_h acting on $H^{k-1,\gamma}$ 1-forms is injective for all $\gamma < -1$ whenever h is sufficiently close to zero in the $H^{k,\beta}$ norm. The uniform injectivity follows in the standard way from the injectivity of each Q_h and the fact that the constant in the elliptic estimate for these operators is independent of h, again provided h is sufficiently near to 0.

Image of Q_0

The $\phi\lambda$ term in (25) was specifically chosen in Section 3.5 to satisfy the integral condition $\int_{\mathbb{R}^3} \langle \lambda\phi, \mathrm{d}x^b \rangle \neq 0$ (since $\lambda_a \neq 0$ for all a). This condition ensures that indeed $2\phi\lambda_a$ is not in the image of the operator $Q_0 = \Delta_\delta$ acting on the space of $H^{l,\gamma}$ 1-forms of $\mathbb{R}^3$ because the image of Δ_δ in $H^{l,\gamma}$ for $\gamma < -1$ is perpendicular to the harmonic polynomials of degree less than the nearest integer less than γ, and this always includes the constants.

The Stability Lemma thus applies to equation (26) and implies that $\phi\lambda$ can not be in the image of Q_h when h is sufficiently small in the $H^{k,\beta}$ norm, unless of course $\lambda = 0$. Now, by the injectivity of the operator Q_h, this in turn implies that $\|\Gamma\| = 0$, or that $\Gamma^a = 0$ for each a. Consequently, the harmonic coordinate condition for the metric $\delta + h$ is satisfied, and as indicated earlier, this implies that the the metric $\delta + h$ and the tensor $S(h, X, T)$ satisfy the time-symmetric extended constraint equations (16). This completes the proof of the Main Theorem. □

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